MATHEMATICAL lens

Dave Ebert

Playground Icosahedron

In spring 2007, the Lions Club of McFarland, Wisconsin, built a unique playground in Lewis Park that includes the climbing structure seen in **photograph 1**. This structure is made up of a number of metal bars of the same length that form equilateral triangular regions. The play system was supplied by Lee Recreation in Cambridge, Wisconsin, and built by community volunteers.

Photograph 1
A net for climbing in Lewis Park, McFarland, Wisconsin

Photograph 2
A truncated icosahedron within the icosahedron

DAVE EBERT

1. (*a*) Lee Recreation sells three versions of the climbing structure shown in **photograph 1**. The length of the metal bar used to make each size and the cost per structure are provided in **table 1**. Graph the cost versus the length and develop an algebraic model that describes a possible relationship between the cost and the length.

   (*b*) Use your algebraic model from (*a*) to predict the cost of a structure for which the length of the metal bar is 150 inches.

   (*c*) Use your algebraic model from (*a*) to predict the length of a metal bar of a structure that sells for $40,000.

2. Ebert measured the distance between the nodes and found it to be 84 inches. The length of the metal bar for the structure that the Lions Club purchased is 75.5 inches. Why are these lengths not the same?

3. If the icosahedron were completely covered with material, how much material would be needed? In other

“Mathematical Lens” uses photographs as a springboard for mathematical inquiry. The goal of this department is to encourage readers to see patterns and relationships that they can think about and extend in a mathematically playful way.

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Table 1

Lee Recreation Climbing Structures

Size of Structure	Length of Metal Bar (x)	Cost of Structure (y)
Small	54.5 inches	$14,999
Medium	75.5 inches	$19,499
Large	107 inches	$27,499

Table 2

Euler's Formula

	Vertices (V)	Faces (F)	Edges (E)
Tetrahedron			
Cube			
Octahedron			
Dodecahedron			
Icosahedron			

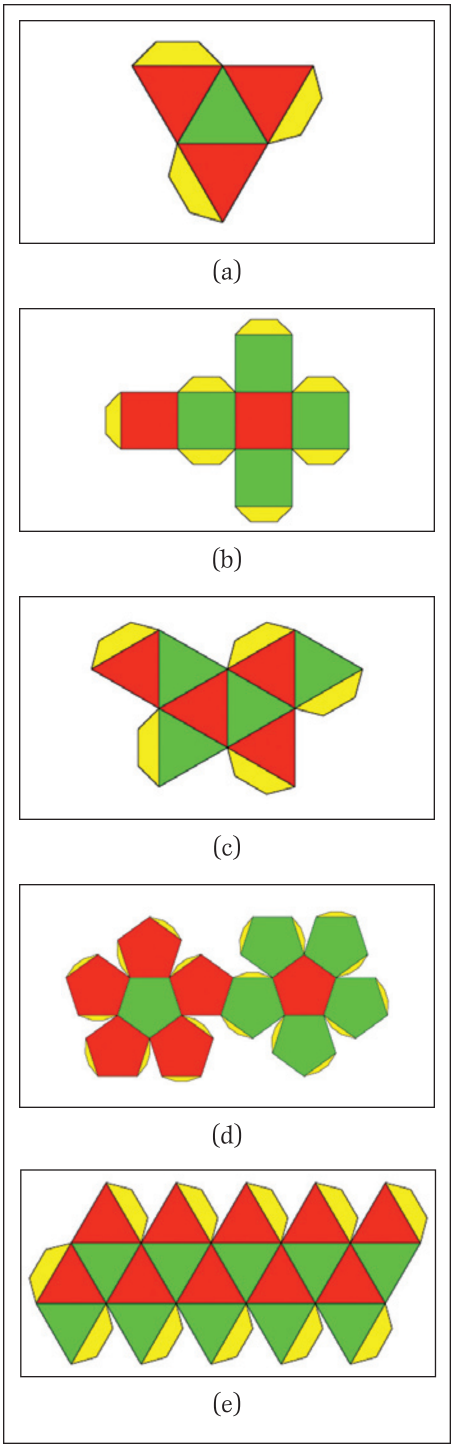

Fig. 1 These nets can be used to make the five platonic solids: a regular tetrahedron (**a**), a cube (**b**), a regular octahedron (**c**), a regular dodecahedron (**d**), and a regular icosahedron (**e**).

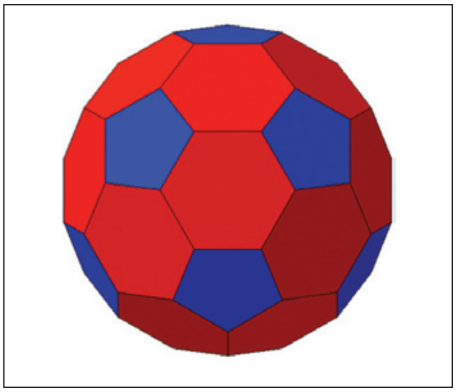

Fig. 2 A truncated icosahedron can be created by using Pedagoguery Software Inc.'s Poly (www.peda.com/poly).

words, what is the surface area of the icosahedron?

4. (*a*) Use the nets in **figure 1** to make a regular tetrahedron, a cube, a regular octahedron, a regular dodecahedron, and a regular icosahedron, respectively. Count the number of vertices (V), faces (F), and edges (E) for these five solids and record your results in **table 2**. Try to determine a relationship among V, F, and E.

(*b*) The relationship among the number of vertices, faces, and edges may not be obvious. A strategy that can be used to discover this relationship is to assume that it is of the form $aV + bF + cE = 1$ (in other words, each quantity is a linear function of the other two). Note that no loss in generality results from using the number 1 in this equation. Any number could be used, but then we could divide both sides of the equation by this number and relabel the unknowns as p, q, and r, an approach that puts us right back at where we started. Select three cases from **table 2**, set up three equations in three unknowns, and solve for a, b, and c.

(*c*) Test the validity of the formula from question 4(*b*) by seeing whether it works for the remaining two cases in **table 2**. If the formula also works for these cases, does this fact constitute a proof that the formula works for all solids?

5. Inside the metal icosahedron structure in **photograph 1** is an additional climbing structure made of rope. A regular pentagon extends from each vertex of the metal icosahedron. These pentagons are then connected to one another to make a number of regular hexagons (see **photograph 2**). The resulting figure, which looks like a soccer ball, is a truncated icosahedron (see **fig. 2**). Show that the relationship determined in question 4(*b*) continues to hold for the truncated icosahedron.

MATHEMATICAL LENS solutions

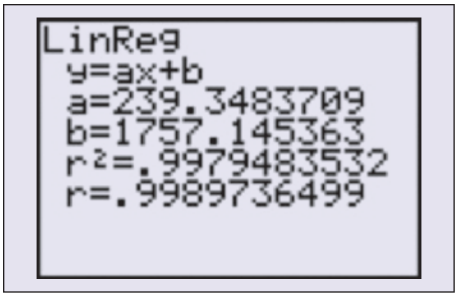

Fig. 3 The calculator obtains a linear relationship between bar length and cost.

1. (*a*) You can use the TI-84 to study the relationship between the cost per structure and the length of the metal bar. Begin your analysis by entering the data in **table 1** into the List Editor. The relationship appears to be linear, and the results for the regression line (see **fig. 3**) show that there is a strong positive correlation between the two quantities.

Table 3

Euler's Formula

	Vertices (V)	Faces (F)	Edges (E)
Tetrahedron	4	4	6
Cube	8	6	12
Octahedron	6	8	12
Dodecahedron	20	12	30
Icosahedron	12	20	30

(*b*) Using the line of best fit and the TI-84, show that when $x = 150$, then $y = \$37,659.40$. Therefore, the cost of the structure would be about \$37,659.00.

(*c*) The value of x for which $y =$ \$40,000 can be found in many ways: (1) by solving a linear equation by hand; (2) by using the Solver capability of the TI-84; or (3) by finding the point of intersection between the line of best fit and $y = \$40,000$. Using the first option, we have $239.3483709x + 1757.145363 = \$40,000.00$ for x yields $x \approx 160$ inches.

2. The difference in lengths can be accounted for by the thickness of the caps that the metal bars slide into.

3. The surface of the icosahedron consists of 20 equilateral triangles. The area of an equilateral triangle with side length s is equal to

$$\frac{\sqrt{3}}{4}s^2.$$

The surface area of the icosahedron is equal to

$$20 \text{ in} \times \frac{\sqrt{3}}{4} \text{in} \times \left(84 \text{ in}\right)^2 \approx 61106.75 \text{ in}^2.$$

Therefore, the surface area is approximately equal to 61,106 in^2, or about 424 ft^2.

4. (*a*) The completed **table 2** should look like **table 3**.

(*b*) Using the first three cases in **table 3**, you can set up three equations in three unknowns as follows:

$$4a + 4b + 6c = 1$$
$$8a + 6b + 12c = 1$$
$$6a + 8b + 12c = 1$$

The solution to this system of equations is $a = 1/2$, $b = 1/2$, and $c = -1/2$. Therefore, the relationship among V, F, and E appears to be $V + F - E = 2$.

(*c*) The formula is valid for the dodecahedron and the icosahedron. It cannot be concluded that the formula is valid for all other solids. A finite number of cases is not enough to constitute a proof.

5. The truncated icosahedron has 60 vertices, 32 faces, and 90 edges. Note that $V + F - E = 60 + 32 - 90 = 2$. Interested readers might want to try their hand at proving this formula, commonly called Euler's formula.

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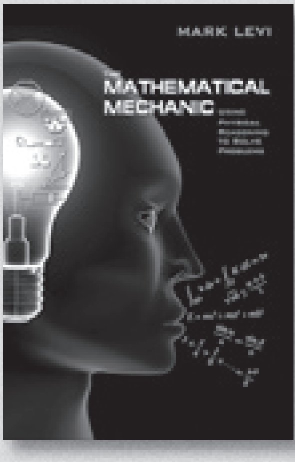
For a mathematical photograph for which you may create your own questions, go to the NCTM Web site: www.nctm.org/mt. Send your questions to the "Mathematical Lens" editors.



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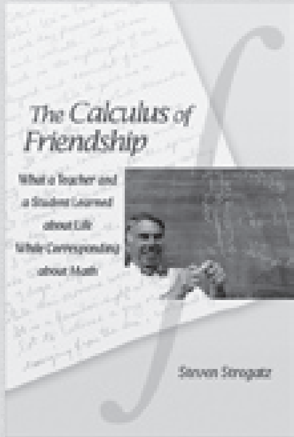

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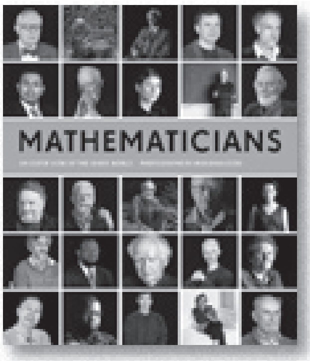

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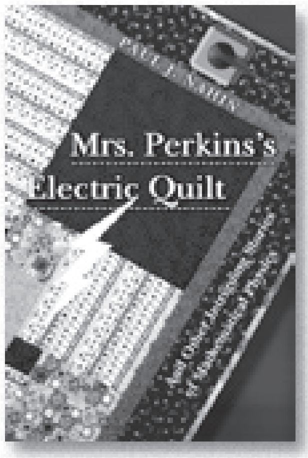

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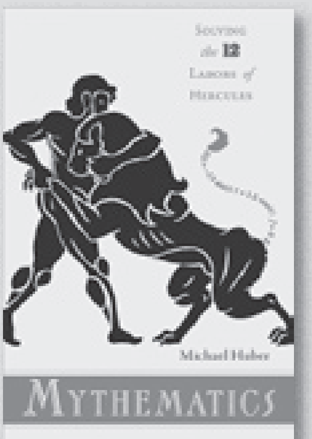

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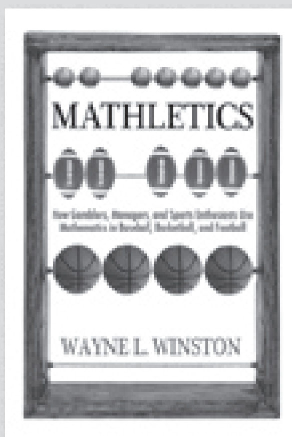

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MARTHA LOWTHER

"Climbing Forever with No End in Sight," photograph taken at Indian Beach, North Sydney, Nova Scotia, by Martha Lowther, who teaches mathematics at the Tatnall School, Wilmington, Delaware